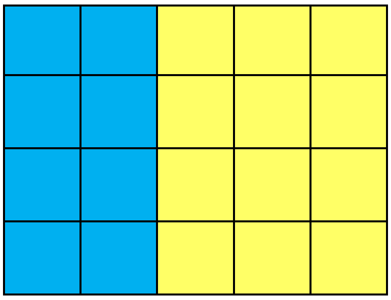
Distributivity



1. Distributivity – the concept

Here we count 4×5 squares by adding 4×2 dark squares to 4×3 light squares. So

$$4 \times 5 = 4 \times (2 + 3) = 4 \times 2 + 4 \times 3,$$

$4 \times \mathbf{5}$ gets distributed to 2 and 3. We can also work backwards to count the **dark** squares:

$$4 \times 2 = 4 \times (5 - 3) = 4 \times 5 - 4 \times 3.$$

$4 \times \mathbf{2}$ gets distributed to 5 and -3.
Clearly the same counting argument works if we replace 4, 2, 3 by any other numbers. Let's take three unknown numbers and call them a, b and c.

The Distributivity Law:

$$a(b + c) = ab + ac = (b + c)a.$$

Note: When working with letters, it is customary to leave out the $\times$ sign, especially as it can be confused with x. So $a(b + c)$ means $a \times (b + c)$ and ab means $a \times b$.

Note: we also checked $\quad a(b - c) = ab - ac = (b - c)a \quad$ for any positive difference.

2. Mental Arithmetic: $\times 99$ and $\times 999$.

Here's a trick to do multiplication quicker:.
a) Examples:
i) $68 \times 99 = 68 \times (100 - 1) = 68 \times 100 - 68 \times 1 = 6800 - 68$ = 6732 and
ii) $637 \times 999 = 637 \times (1{,}000 - 1) = 637 \times 1{,}000 - 637 \times 1 = 637{,}000 - 637$
$= 636{,}363.$

b) Practice exercises:

i) $500 - 25$ ii) $4{,}000 - 324$
iii) 25×99 vii) 324×999

c) Try these in your head:

i) $333{,}333 \times 11$ ii) $232{,}323 \times 11$ iii) $999{,}999 \times 7$

d) Quick multiplication: try these in your head or on paper but no long multiplication:

i) 75×99 ii) 78×99 iii) 625×999 iv) 629×999

e) Use a distributivity trick to solve these $\times 18$ and $\times 198$ exercises:

i) 65×18 ii) 335×18 iii) 128×198 iv) 154×198

1. Choose any four natural numbers, call them a, b, c, d. In each of the following cases, draw a coloured squares diagram that explains the formula. like how distributivity is explained above.
a) $a(b + c + d) = ab + ac + ad.$
b) $(a + b)(c + d) = ac + ad + bc + bd.$
c) $(a + b)^2 = a^2 + 2ab + b^2.$
Hint: $(a + b)$^2 $= (a + b)(a + b)$

3. More Mental Arithmetic: Difference of two squares.

Take any two numbers, call them x and y. Then the product between their difference and their sum is:

$$(x-y)(x+y) = x^2 - y^2$$

Proof: $(x-y)(x+y) = (x-y)x + (x-y)y = xx - yx + xy - yy = x^2 - y^2$.

c) Example: We want to find 59x61. $59 = 60 - 1$ and $61 = 60 + 1$ so we can apply the difference of squares formula for $x = 60$ and $y = 1$.

$$59 \times 61 = (60-1)(60+1) = 60^2 - 1 = 3600 - 1 = 3599$$

d) Practice exercises:

Try to do these in your head, or on paper, but with no long multiplication:

i) 47×53 ii) 65×75

i) 39×41 ii) 28×32 iii) 67×73 iv) 46×54 v) $73 \times$

4. More Arithmetic Tricks:

Sometimes it pays to pay attention to repeating terms in long calculation. For example, if a number a comes up in two different products ab and $-ac$, then writing $ab - ac$ like $a(b-c)$ can simplify things considerably.

Example: Find A and B without using any calculators or long multiplication:

$E = 3333 \times 6543 + 2222 \times 6667 - 3333 \times 4321.$

$F = 499 \times 1112 - 1111 \times 399.$

$A = 5999 \times 2001 - 2001 \times 2999 + 3000 \times 1999 - 5999.$

$B = 4848 \times 63 + 693 + 63 \times 5252.$

$C = 171 \times 54 - 53 \times 161.$

$D = 1995 \times 1998 - 1997 \times 1996.$

5. Did you know? Products of negative numbers and distributivity.

It's no secret that $2 \times (-3) = (-3) + (-3) = -6$.

But why is it that $(-2) \times (-3) = 6$?

Answer: Distributivity! Once people noticed the property

$$a(b+c) = ab + ac$$

for all positive numbers, they decided that the negative numbers must abide by it as well. When trying to calculate $(-2) \times (-3)$, they first thought about the meaning of (-2). They remembered that (-2) was defined by the condition

$$(-2) + 2 = 0.$$

From here, all they had to do was multiply the above equation by (-3) and use distributivity:

$$(-2) \times (-3) + 2 \times (-3) = \left((-2) + 2\right) \times (-3) = 0 \times (-3) = 0.$$

But $2 \times (-3) = -6$ so the equation above says $(-2) \times (-3) - 6 = 0$, that is

$$(-2) \times (-3) = 6.$$

In fact, the same proof works for any positive numbers a and c:

$$(-a) \times (-c) = ac .$$

Use these rules, or directly the definition of $-c$, to explain why the following properties are true for all numbers a and b:

a) $-(-a) = a$.

b) $-(a+b) = -a - b$.

c) $-(a-b) = -a + b$

6. Did you know? Long multiplication/division and distributivity.

The Distributivity Law is the reason behind the long multiplication and long division algorithms.

Examples:

a) Long multiplication for 23 × 761:

First 761 is split from right to left: 761 =1+60+700. Then by distributivity:

$$23 \times 761 = 23 \times 1 + 23 \times 60 + 23 \times 700$$
$$= 23 + 1380 + 16100$$

The long multiplication is a particular arrangement of the numbers with alignment by the last digit, like this:

```
      2 3
  × 7 6 1
  --------
      2 3
  1 3 8 0
1 6 1 0 0
---------
1 7 5 0 3
```

where, in the first row, 23 = 23 × 1, in the second row, 1380 = 23 × 60, and, in the third row, 16100 = 23 × 700. In time, people have tired of writing the trailing zeros that are due to the powers of 10, and now remember them just by the placement of the other digits.

(Copied from : http://www.cut-the-knot.org/Curriculum/Arithmetic/LongMultiplication.shtml Check the website for a nice Java applet and other resources.)

b) Long division is also a consequence of the Distributivity Law:

$$17503 = 16100 + 1380 + 23 = 23 \times 700 + 23 \times 60 + 23 \times 1 = 23 \times 761.$$

Long division splits this into steps by teasing out the hundreds, the tens and then the units:

```
   __761
23 | 17503
     161
      140
      138
        23
        23
         0
```

7. Did you know? Distributivity makes sense of relative speed ...

Activity Sheet – Distributivity

1. Choose any four natural numbers, call them a, b, c, d. In each of the following cases, draw a coloured squares diagram that explains the formula. like we did in the class discussion:
a) $a(b + c + d) = ab + ac + ad$.
b) $(a + b)(c + d) = ac + ad + bc + bd$.
c) $(a + b)^2 = a^2 + 2ab + b^2$.
Hint: $(a + b)$^2 $= (a + b)(a + b)$.

2. a) Try these in your head:
i) $333{,}333 \times 11$ ii) $232{,}323 \times 11$ iii) $999{,}999 \times 7$
b) Quick multiplication: try these in your head or on paper but no long multiplication:
i) 75×99 ii) 78×99 iii) 625×999 iv) 629×999
c) Use a distributivity trick to solve these $\times 18$ and $\times 198$ exercises:
i) 65×18 ii) 335×18 iii) 128×198 iv) 154×198

3. Quick multiplication:
i) 39×41 ii) 28×32 iii) 67×73 iv) 46×54 v) 73×87

4. Find A, B, C and D without using any calculators:
$A = 5999 \times 2001 - 2001 \times 2999 + 3000 \times 1999 - 5999.$
$B = 4848 \times 63 + 693 + 63 \times 5252.$
$C = 171 \times 54 - 53 \times 161.$
$D = 1995 \times 1998 - 1997 \times 1996.$

5. We now know that $-c = (-1) \times c$ and $c = (-1) \times (-c)$ for all numbers c.
Use these rules, or directly the definition of $-c$, to explain why the following properties are true for all numbers a and b:
a) $-(-a) = a$.
b) $-(a + b) = -a - b$.
c) $-(a - b) = -a + b$
Hint: In each case you can take c to be the number in the bracket.
6. Some mystery number problem?
7. Find A and B without using any calculators:
$A = 999 \times 54{,}321 - 45{,}679.$
$B = 2016 \times 2015 - 2015 \times 2014 - 2014 \times 2013 + 2013 \times 2012$

Solutions:

1.

2. a) i)

b) i) 7425 ii) 7722 iii) 624,375 iv) 628,371

c) Write 18=20-2 and 198=200-2:

i) $65 \times 18 = 130 \times 20 - 130 \times 2 = 2600 - 260 = 2340.$

ii) $335 \times 18 = 335 \times 20 - 335 \times 2 = 6700 - 670 = 6030.$

iii) $128 \times 198 = 128 \times 200 - 128 \times 2 = 25{,}600 - 256 = 25{,}344.$

iv) $154 \times 198 = 154 \times 200 - 154 \times 2 = 30{,}800 - 308 = 30{,}492.$

Move on to divisibility:

7. A Mind-Reader computer programme

Play this game a number of times:

http://www.cut-the-knot.org/Curriculum/Magic/MindReaderNine.shtml

Can you explain what's going on?

Hint: Since we can't seriously believe that the computer is a mind reader, it must be that the programme assigns the same shape to all the possible answers to the problem. Play again and check the numbers having the same shape as your solution. Notice any special property? Now try to prove it!

Solution: All possible answers to the problem are divisible by 9.

Let's suppose that the number is written AB, with A the tens digits and B the unit digit. Then the number is equal to $10A + B$ and the number read backwards is $10B + A$. (Here $10A$ means $10 \times A$, similarly $10B$).

The difference between the number and the number read backwards is

$$10A + B - (10B + A)$$
$$= 10A + B - 10B - A$$
$$= 9A - 9B = 9(A - B),$$

Which is always divisible by 9!