# Intriguing Indices

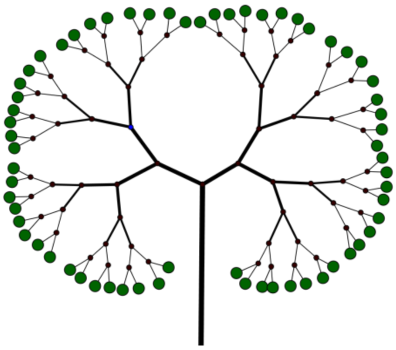
## 1. Double Trouble

Quick counting:

How many leaves are in the tree?

The leaves are the dots at the end.

Can you make a strategy to count the leaves in a clever way?



## 2. Multiplication Rule

a) Calculate: $3^{3n-4} \times 3^{12-4n} \times 3^{n-5}$.

b) Find the smallest positive integer $n$ such that the number $48 \times 54 \times n$ is a perfect power (perfect power can be written as $a^b$ where a and b are integers. Example: $8000 = 20^3 = 4^3 \times 5^3$)

c) Which is larger: $21^{2013}$ or $84 \times 3^{2011} \times 7^{2012}$ ?

## 3. Division Rule

a) Fill in the following table:

| | | | | | $a$ | $a^2$ | $a^3$ | $a^4$ |
|---|---|---|---|---|---|---|---|---|
| -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |

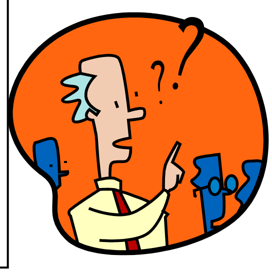
b) Simplify:

$$i)\ \frac{5^{4n+6} \times 7^{n+2}}{5^{n+3}} \times \frac{7^{n+4}}{5^{3n+1} \times 7^{2n+5}}$$

$$ii)\ (5^{2n+1} \div 5^{3n-2}) \times (15^{n+1} \div 3^n)$$

## 4. A guessing game.

This is played in pairs. The 1st player thinks of a number between 1 and 63. The 2nd player tries to guess it. After every guess, the 1st player has to state (truthfully!) whether his/her chosen number is bigger or smaller than the one just heard. This goes on until the mystery number is found. The players take turns being first and each time, the number of trials is tallied. The player who needed fewer trials wins.



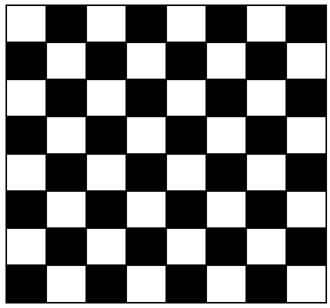
a) Play the game 3 times with your partner – what is a good initial guess?
b) What is a good second guess?
c) Can you think of a strategy? How many moves do you need to win?
d) How many moves do you need to win if now the number can be between 1 and 1000?

**5. Guessing game 2D**

Two chess players on a break play a guessing game.

The 1st player places an imaginary rook on one of the squares.

The 2nd player touches any point on the board. If the point is inside the same square as the imaginary rook, or on its border, the game is over. Otherwise, the 1st player indicates the relative position of the rook by UP/DOWN and LEFT/RIGHT.



a) What's the largest possible number of "moves" in this game?

b) The same question assuming the board was made out of 4096 small black-and-white squares.

**6. Multiple Indices:**

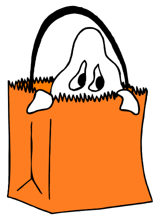
*a)* Is this number a perfect square ( A perfect square is a number that can be written as $b^2$ for some number b) :

$$\frac{81^{n+2}\times 4^{2n+3}}{6^{5-4n}\times 54}?$$

$b$) Find $n$ such that: $3^{6n+12}+9^{3n+6}+27^{2n+4}=81.$

**7. Trick-or-Treat**

Which of the two numbers is larger?



a) $2^{3^2}$ or $(2^4)^2$ ?

b) $\sqrt[4]{\left(\frac{1}{2}\right)^{2^{2^2}}}$ or $\sqrt{\left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)^2}$ ?

c) $2^{2^{2^2}}$ or $3\times(3^5)^3$ ?

d) $5\times 5^{-7}$ or $\frac{1}{2}\times 2^{-2}\times 3^{-3}\times(6^{-3})$ ?

e) $7^{25}$ or $2^{75}$ ?

f) $4^{30}\times 3^{20}$ or $5^{40}$ ?

**8. Sums of exponents**

a) Solve for $n$:

$$3^{n+3}-2\times 3^{n+2}+3^{n+1}=972.$$

*b)* Show that $4^{2+n}-2^{2+2n}+4^n$ is always divisible by 13.

*c)* Show that $8^{n+2}-4^{n+1}\times 2^{n+2}+2^{3n}+2\times 5^{n+2}-5^n$ is always divisible by 7.

**8.** How many digits does the number $4^{16}\times 5^{25}$ have? (Hint: How many digits does $10^{14}$ have?)

**9. Digit Tricks**

*a*) Find the last digit of $5^{555}$

b) Find the last digit of $3^{2012}$

c) Find the last digit of $9^{10^{100}}$

d) Find the last digit of $2^{2^{2012}+2}$

e) The number $N=1000!=1000\times\ 999\times\ 998\times\ \cdots\times\ 3\times\ 2\times\ 1$ ends in exactly how many zeroes?

# Intriguing Indices – Practice Questions

1. Simplify:

$a)\ 27^{6-2n} \times 3^{4n-1} \times 9^{n-1}$ $\quad b)\ \dfrac{4^{4k+1} \times 2^{k+2}}{8^{3k}}$ $\quad c)\ \dfrac{2^{-1} + 2^{0} + 2^{1}}{2^{-4} + 2^{-3} + 2^{-2}}$

**2.** Solve for $a$:

$$(7^{3a+1} \times 3^{5a+6}) \div (3^{2a+4} \times 21^{2a+1}) = 189$$

**3.** Solve for $x$ when $(a \neq 0)$:

$$a) a^{x^2} = 1 \quad b) a^{x^2} = a \quad c) a^{x^3} = a$$

**4.** Find the largest integer $n$ such that $n^{200} < 5^{300}$.

**5.** Compare each of these pairs of numbers

$a)\ (12^{100},\quad 7 \times 3^{99} \times 2^{199})$ $\quad b)\ (\dfrac{6^{6^2}}{6^2 \times 6^2},\quad \left(\left(2^{2^{2^2}}\right)^2\right)^2)$

$c)\ (4^{30} \times 3^{20}, 5^{40})$ $\quad d)(5^{66}, 3^{99})$ $\quad e)(2^{6018}, 3^{4012})$

**6.** a) Write $5^{11}$ as the sum of 2 squares.
b) Write $17^{17}$ as the sum of 2 squares.
c) Write $126^{301}$ as the sum of 2 cubes.
d) Write $10^{511}$ as a sum of two square numbers.
e) Write $9^{2014}$ as a sum of two cube numbers.

**7.** a) Calculate $1^2 + 5^2 + 10^2 + 27^2 + 34^2$.
b) Write $2011^{2011}$ as a sum of five perfect squares.

**8.** How many digits does the number $4^{16} \times 5^{25}$ have?

**9. Digit Tricks**
*a*) Find the last digit of $5^{555}$

b) Find the last digit of $3^{2012}$

c) Find the last digit of $9^{10^{100}}$

d) Find the last digit of $2^{2^{2012}+2}$

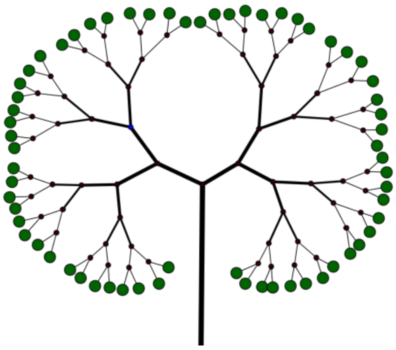
e) The number $N = 1000! = 1000 \times 999 \times 998 \times \cdots \times 3 \times 2 \times 1$ ends in exactly how many zeroes?

## Intriguing Indices – Solutions and Class Discussion

### 1. Double Trouble

Quick counting:

How many leaves are in the tree?



***Solution and class discussion:*** It is not hard to conclude that there are 64 leaves, but it is interesting to discuss the different ways to reach this number:

i) The tree can be split in 2 equal parts by left-right symmetry. The leaves in each part are roughly arranged in 4 groups of 8 leaves each:

$$2 \times 4 \times 8 = 64$$

ii) Starting from the tree trunk, count how many branching points there are en route to a leaf. Each branching in 2 doubles the number of leaves. There are 6 branching points so

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$$

leaves in total. One can also place a number at each branching point forming this sequence: 1,2,4,8,16,32,64.

Here 6 is an *index* or an *exponent*. If we are calling it an index, we must be careful as there are other things in maths referred to as indices as well. 64 is called a *power* of 2 because we can write it as a product of 2s *only.*

All the numbers above are powers of 2: indeed $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $64 = 2^6$. The first expression above becomes:

$$2 \times 4 \times 8 = 2^1 \times 2^2 \times 2^3 = 2^6.$$

Do you notice anything interesting about the indices? They sum as $1 + 2 + 3 = 6$.
And no wonder: $2 \times 4 \times 8 = 2 \times (2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$.
In both cases, 2 is multiplied by itself 6 times.

**Multiplication Rule**

$$a^m \times a^n = a^{m+n}$$

Explanation: You are multiplying $a$ by itself $m$ times and then $n$ times. So overall you repeated the same action $m + n$ times.

**2. Multiplication Rule**

a) Calculate: $3^{3n-4} \times 3^{12-4n} \times 3^{n-5}$.
b) Find the smallest positive integer n such that the number $48 \times 54 \times n$ is a perfect power (that is, the power of some integer).
c) Which is larger: $21^{2013}$ or $84 \times 3^{2011} \times 7^{2012}$ ?

*Solution: a)* $3^{3n-4+12-4n+n-5} = 3^{3n-4n+n-4+12-5} = 3^3 = 27$.
b) Hint: write each of the numbers 48 and 54 as products of powers and then combine:
$48 \times 54 \times n = 3 \times 2^4 \times 2 \times 3^3 \times n = 3^4 \times 2^5 \times n$ note that the order of the factors in the product doesn't matter (*commutativity of multiplication*).
You can insert a factor of 3 to $3^4 \times 2^5$ to get $3^5 \times 2^5 = (3 \times 2)^5 = 6^5$ = a perfect power.
However, you can do even better if you notice $3^4 = 3 \times 3 \times 3 \times 3 = 9 \times 9 = 9^2$ and $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 8^2$ so you can insert a factor of 2 to get

$$3^4 \times 2^5 \times 2 = 3^4 \times 2^6 = 9^2 \times 8^2 = 72^2.$$

The smallest number we can find is $n = 2$.
While solving this problem you may have noticed $(a^n)^p = a^{np}$ because $np$ copies of $a$ can be grouped into $p$ copies of $a^n = (a \times a \times \ldots \times a)$.

Can you explain why $(ab)^n = a^n \times b^n$?
$(ab)^n = ab \times ab \times ab \times \ldots \times ab = a \times a \times a \times \ldots \times a \times b \times b \times b \times \ldots \times b = a^n \times b^n$

*c)* $21^{2013} = 3^{2013} \times 7^{2013} = 3 \times 3^{2012} \times 7^{2013}$ .
$84 \times 3^{2011} \times 7^{2012} = 2^2 \times 3 \times 7 \times 3^{2011} \times 7^{2012} = 4 \times 3^{2012} \times 7^{2013}$.
So the second number is bigger.

***3.* Division Rule**

a) Fill in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | $a$ | $a^2$ | $a^3$ | $a^4$ |
| -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |

b) Simplify:

$i)\ \dfrac{5^{4n+6} \times 7^{n+2}}{5^{n+3}} \times \dfrac{7^{n+4}}{5^{3n+1} \times 7^{2n+5}}$ $\qquad ii)\ (5^{2n+1} \div 5^{3n-2}) \times (15^{n+1} \div 3^n)$

*Solution:* Let's discuss the pattern:

| $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ | … |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | … |

Above, we have drawn a table where we keep multiplying $a$ by itself and keep count of how many times we have on the bottom. If we take any term on the top line and multiply it by $a$, we notice that we move one place to the right, which is the same as adding one on to the index. We could extend this to show the multiplication rule! What if we divide a term by $a$? Well there will now be one less $a$ in the product, so it will shift to the $left$ by one place. We subtract one from the index. What if we divide by $a^2$? We will move two places to the left, subtracting two from the index. If we divide by $a^n$, we will move $n$ places to the left, subtracting $n$ from the index. So

**Division Rule**

$$\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$$

Hence $\boldsymbol{a^0} = \mathbf{1}$. We can look at this in different ways. Starting at $a^n$, let's multiply it by $a^0$ and see where it takes us on the chart. $a^n \times a^0 = a^{n+0} = a^n$. So multiplying by $a^0$ doesn't change the number at all. But there is only one number that does this, 1! Hence $a^0 = 1$. Another way to explain this is to apply the formula $\frac{a^m}{a^n} = a^{m-n}$ in the case when $m = n$. So $a^0 = 1$. And this is true for all $a$... or is it?

From above, we would say that $0^0 = 1$. But $0^1 = 0$, $0^2 = 0$ and it's fairly easy to see that $0^n = 0$. But then shouldn't $0^0 = 0$? So what's wrong? This is an inconsistency in arithmetic: We say that $0^0$ is undefined, there is no number that represents it because it doesn't obey all rules the way we would like it to. This happens sometimes in mathematics, for example what is $\frac{0}{0}$? Note that this is exactly why $\frac{a^m}{a^m} = a^{m-m} = a^0$ doesn't work for 0.

**Negative exponents**

Assuming that $a \neq 0$, we can fill in the table as below: To get to negative exponent, we need to move to the left of the chart, and this involves division. So if we start at a number like $a^3$, and keep moving left by dividing we see

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ... | $\frac{1}{a^3}$ | $\frac{1}{a^2} = a^{-2}$ | $\frac{1}{a} = a^{-1}$ | 1 | $a^1$ | $a^2$ | $a^3$ | ... |
| ... | -3 | -2 | -1 | 0 | 1 | 2 | 3 | ... |

And we can keep moving back like this. Thus

$$a^{-n} = \frac{1}{a^n}$$

Another way to see this is to take $m = 0$ in the formula $\frac{a^m}{a^n} = a^{m-n}$.

$b)\ i)$ 175; $ii)$ $5^{2n+1-(3n-2)+n+1} \times 3^{n+1-n} = 5^4 \times 3$ =1,875

**4. A guessing game.**

Play this in pairs. The 1st player thinks of a number between 1 and 63. The 2nd player tries to guess it. After every guess, the 1st player has to state (truthfully!) whether his/her chosen number is bigger or smaller than the one just heard. This goes on until the mystery number is found.

The players take turns being first and each time, the number of guesses is tallied.

The player who needed fewer guesses wins.

?

a) If you play well, you should be able to guess your number in less than            moves. Fill in the smallest number that you think will always work. Try it out!

b) As 2nd player, I'll use my minimal-guess strategy demonstrated in class. What strategy would you choose to stall me as much as possible?

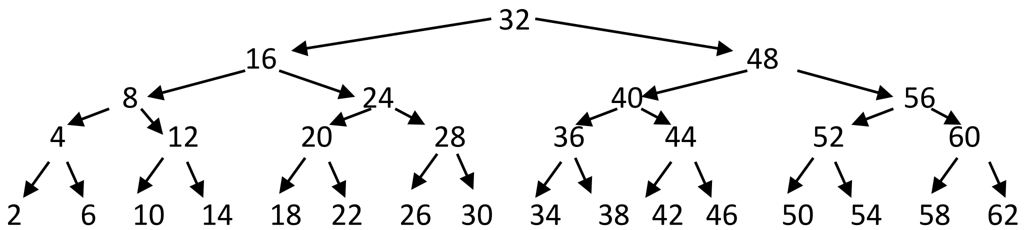
c) At most how many tries would you need to guess a number between 1 and 1000?

*Solution:* After the pairs have played, have a class session with a volunteer student as 1st player and the lecturer/tutors as 2nd player. The volunteer can write the mystery number on a piece of paper and circulate it to the class.
Here is the 2nd player's strategy:
Step 1: Guess 32.
Step 2: Add or subtract 16 to your previous guess: "bigger"= + and "smaller" = −
Step 3: Add or subtract 8 to your previous guess.
Step 4: Add or subtract 4 to your previous guess.
Step 5: Add or subtract 2 to your previous guess.
Step 6: Add or subtract 1 to your previous guess.
Less than 7 moves (at most 6 moves) are needed because $63 = 2^6 - 1$.
Also note that $32 + 16 + 8 + 4 + 2 + 1 = 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1 = 2^6 - 1 = 63$

$$32 - 16 - 8 - 4 - 2 - 1 = 2^5 - (2^4 + 2^3 + 2^2 + 2 + 1) = 1.$$

So if we used this strategy, we'd need 7 moves to get to 64.

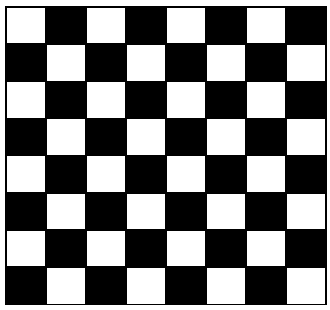
b) Here is the 2nd player's cheat sheet for the first 5 steps:




It's pretty clear that all the odd numbers will fall in the last step. How can we prove this? We'll return to this after discussing binary basis.

c) $1000 < 1024 = 2^{10}$ so 10 guesses should suffice. One may introduce the logarithm notation: $2^{10} = 1024$ is equivalent to $10 = \log_2 1024$.
In general, $\log_2 a$ answers the question: 2 raised to which power equals $a$?
The number of trials we'll need to guess a number between 1 and $a$ is at most $\log_2 a + 1$.

**5. Guessing game 2D**



Two chess players on a break play a guessing game.
The 1st player places an imaginary rook on one of the squares.
The 2nd player touches any point on the board. If the point is inside the same square as the imaginary rook, or on its border, the game is over. Otherwise, the 1st player indicates the relative position of the rook by UP/DOWN and LEFT/RIGHT.
a) What's the largest possible number of "moves" in this game?
b) The same question assuming the board was made out of 4096 small black-and-white squares.

*Solution:* a) This is a very short game indeed, with only 3 moves needed. We first divide the board into 4 equal $4 \times 4$ squares and touch the point in the centre, then move to the $4 \times 4$ square indicated by the UP/DOWN and LEFT/RIGHT hints, divide this into 4 equal $2 \times 2$ squares, move on and again divide by 4 til at the last step we get $1 \times 1$ squares.
Looking at the total number of small square this is explained by $4^3 = 64$. Looking at side lengths: $64 = 8^2 = (2^3)^2$. Perhaps this is a good moment to note that

$$(2^2)^3 = (2^3)^2 = 2^6.$$

So 64 is both a *perfect square* and a *perfect cube*.

**Multiple Indices Rule:**

$$(a^m)^n = a^{m \times n}$$

Indeed, from our definition of exponents

$$(a^m)^n = \underbrace{a^m \times a^m \times \ldots \times a^m}_{n \text{ times}} = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{m \text{ times}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ times}} \times \ldots \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ times}}}_{n \text{ times}}$$

$$= a^{mn}$$

b) $4096 = 4 \times 1024 = 4 \times 2^{10} = 4 \times (2^2)^5 = 4 \times 4^5 = 4^6$ so we will need 6 moves to finish the game.

**6. Multiple Indices:**

*a)* Is this number a perfect square:

$$\frac{81^{n+2} \times 4^{2n+3}}{6^{5-4n} \times 54}?$$

$b)$ Find $n$ such that: $3^{6n+12} + 9^{3n+6} + 27^{2n+4} = 81.$

*Hints:* Write everything in terms of powers of 3 and 2.

*Solution:* $a)$ $3^{4n+8} \times 2^{4n+6} \times (3 \times 2)^{-(5-4n)} \times 3^{-3} \times 2^{-1} = 3^{4n+8-5+4n-3} \times 2^{4n+6-5+4n-1}$

$$= 3^8 \times 2^8$$

c) By playing with the exponents, we get the equation $6n + 13 = 4$ so $n = -\frac{3}{2}$.
A negative fraction. Ugh.

What is $a^{\frac{1}{2}}$? Can you explain why? Does $a^{\frac{1}{2}}$ always exist?

What if we get rid of the annoying denominator?

$$\left(a^{\frac{1}{2}}\right)^2 = a^{\frac{1}{2} \times 2} = a^1 = a$$

So if we square $a^{\frac{1}{2}}$, we will get $a$, so that means that

$$a^{\frac{1}{2}} = \sqrt{a}$$

Note that this entire argument goes awry if $a < 0$. Indeed, then how could a negative number$a$ equal a positive $\left(a^{\frac{1}{2}}\right)^2$ ? So $a^{\frac{1}{2}}$ is not defined when $a < 0$!

What about $a^{\frac{1}{n}}$? If $a \geq 0$ then by raising it to the $n$-th power we see that

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

**7. Treat-or-Trick**

You are offered a choice of one of two boxes. Each box carries a label stating the number (or fraction!) of identical goodies inside. Compare the labels and pick one:

a) $2^{3^2}$ or $(2^4)^2$ ?

b) $\sqrt[4]{\left(\frac{1}{2}\right)^{2^{2^2}}}$ or $\sqrt{\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2}$ ?

c) $2^{2^{2^2}}$ or $3 \times (3^5)^3$ ?

d) $5 \times 5^{-7}$ or $\frac{1}{2} \times 2^{-2} \times 3^{-3} \times (6^{-3})$ ?

e) $7^{25}$ or $2^{75}$ ?

f) $4^{30} \times 3^{20}$ or $5^{40}$ ?

*a)* The emphasis here is for the students to realise that if two numbers greater than 1 have the same bases but different indices, we can compare the results by comparing the indices:

To simplify $2^{3^2}$, we start from the top and work down. $3^2 = 9$ so

$$2^{3^2} = 2^9$$

We can do this for bigger and bigger "telescopes" of exponents, as long as we start from the top and work down. On the other hand, $(2^4)^2 = 2^8$ so those who like Halloween stuff would choose the 1st box as $2^9 = 2 \times 2^8 > 2^8$.

The general rule:

If the basis $a > 1$, then the number with the larger index is the larger

$$a^m > a^n \text{ for } m > n \text{ whenever } a > 1.$$

because $a^m = a^n \times a^{m-n} > a^n \times 1$.

b) This is weird: you're getting fractions of the goodies:

$$\sqrt[4]{\left(\frac{1}{2}\right)^{2^{2^2}}} = \sqrt[4]{\left(\frac{1}{2}\right)^{16}} = \left(\frac{1}{2}\right)^{16\div 4} = \left(\frac{1}{2}\right)^4 = 2^{-4}$$

$$\sqrt{\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^6} = \left(\frac{1}{2}\right)^{6\div 2} = \left(\frac{1}{2}\right)^3 = 2^{-3}$$

And $-4 < -3$ so the second number is larger. This can also be seen in the following way:

$$\left(\frac{1}{2}\right)^4 = \frac{1}{2} \times \left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^3$$

Because half of a positive quantity is less than the whole.

So for numbers between 0 and 1, the situation is reversed:

$$a^m < a^n \text{ for } m > n \text{ whenever } 0 < a < 1$$

because $a^m = a^n \times a^{m-n} < a^n \times 1$.

*c)* This reduces to $2^{16} < 3^{16}$. If two numbers have the same exponents but different bases (the thing we multiply by itself), then will the number with the bigger base always be bigger?

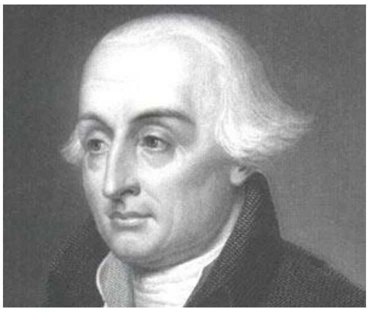
d) This reduces to $5^{-6}$ versus $6^{-6}$. Even though $-6$ is the same on both sides and $5 < 6$, we have $5^{-6} = \left(\frac{1}{5}\right)^6 > \left(\frac{1}{6}\right)^6$ because 1 fifth of something is more than 1 sixth.

To recap, one always has to be careful when negative signs are parts of inequalities... There's also the question as to the actual content of the boxes: are these fractions of a mountain of candy? Then they may be worth something...

e) The idea for both questions is to make either the index or base the same, and then compare using ideas we have built up already. Notice that $2^{75} = (2^3)^{25} = 8^{25} > 7^{25}$.

f) Observing that 10 divides each of the indices, we write the 1st number as $(4^3 \times 3^2)^{10}$ and the 2nd as $(5^4)^{10}$. As $4^3 \times 3^2 = 64 \times 9 = 640 - 64 < 625 = 5^4$, the 2nd number is bigger.

## 8. Lagrange's four-square theorem



**Every natural number can be written as a sum of four squares.**

a) Check that $2013 = 2^2 + 3^2 + 20^2 + 40^2$.
b) Write $2013^2$ as a sum of four non-zero squares. Hint: one of them is $2012^2$.
c) Write $2013^3$ as a sum of four squares.
d) Write $2013^4$ as a sum of four squares.
e) Prove that $2013^n$ can be written as a sum of four non-zero squares for all $n$.

*Solution:* a) Calculate.
b) *Hints*: Difference of two squares and factor out a square whenever possible:

$$2013^2 - 2012^2 = (2013 - 2012)(2013 + 2012) = 4025 = 25 \times 161$$
$$= 25(100 + 25 + 36) = 50^2 + 25^2 + 30^2$$

c) *Hints*: You can factor out a square: $2013^3 = 2013^2 \times 2013 = 2013^2(2^2 + 3^2 + 20^2 + 40^2)$.
d) Same trick: $2013^4 = 2013^2 \times 2013^2$ and use b).
e) Same trick: $n = 2k + 1$ or $2k$.
$2013^{2k+1} = 2013^{2k}(2^2 + 3^2 + 20^2 + 40^2)$ and
$2013^{2k} = \left(2013^{k-1}\right)^2(50^2 + 25^2 + 30^2 + 2012^2)$

## 9. Sums of exponents

a) Solve for $n$:

$$3^{n+3} - 2 \times 3^{n+2} + 3^{n+1} = 972$$

*b)* Show that $4^{2+n} - 2^{2+2n} + 4^n$ is always divisible by 13.
*c)* Show that $8^{n+2} - 4^{n+1} \times 2^{n+2} + 2^{3n} + 2 \times 5^{n+2} - 5^n$ is always divisible by 7.

*Solution:* a) Notice that $3^n$ is a factor of each of the terms on the left hand side. So we use distributivity and take it outside.

$$3^n(3^3 - 2 \times 3^2 + 3^1) = 3^n \times 12 = 972$$

Hence $3^n = 81$ and $n = 4$

*b)* First, we make sure that the base is the same, so our expression becomes $2^{4+2n} - 2^{2+2n} + 2^{2n}$. Now we can factor out $2^{2n}$ to get $2^{2n}(2^4 - 2^2 + 1) = 2^{2n} \times 13$. As it is a multiple of 13, $4^{2+n} - 2^{2-2n} + 4^n$ is divisible by 13.

*c)* The problem here is that we can't simplify everything, as we are working with bases of 2 and 5. So we deal with each part separately. In the end, we should have something like

$$49(2^{3n} + 5^n)$$

So the expression is always divisible by 7.

## Supplementary Exercises

**1.** Simplify:

$a)\ 27^{6-2n} \times 3^{4n-1} \times 9^{n-1}$ $\qquad b)\ \dfrac{4^{4k+1} \times 2^{k+2}}{8^{3k}}$ $\qquad c)\ \dfrac{2^{-1} + 2^0 + 2^1}{2^{-4} + 2^{-3} + 2^{-2}}$

*Answer:* a) $3^{15}$ b) $2^4$ c) 8

**2.** Solve for $a$:

$$(7^{3a+1} \times 3^{5a+6}) \div (3^{2a+4} \times 21^{2a+1}) = 189$$

*Answer:* $a = 1$.

**3.** Solve for $x$ when $(a \neq 0)$:

$$a)a^{x^2} = 1 \quad b)a^{x^2} = a \quad c)a^{x^3} = a$$

*Answer:* $a)\ x = 0$ $\qquad b)\ x = 1, x = -1$ $\qquad c)\ x = 1$.

**4.** Find the largest integer $n$ such that $n^{200} < 5^{300}$.

*Solution:*

$(n^2)^{100} < (5^3)^{100}$, so $n^2 < 5^3$. The largest such $n$ is $11$.

**5.** Compare each of these pairs of numbers

$a)\ (12^{100},\quad 7 \times 3^{99} \times 2^{199})$ $b)\ (\frac{6^{6^2}}{6^2 \times 6^2}, \left(\left(2^{2^{2^2}}\right)^2\right)^2)$

$c)\ (4^{30} \times 3^{20}, 5^{40})$ $d)(5^{66}, 3^{99})$ $e)(2^{6018}, 3^{4012})$

*Solution:* a) $12^{100} = 2^{200} \times 3^{100} = 2 \times 2^{199} \times 3 \times 3^{99} = 6 \times 2^{199} \times 3^{99}$. So the second term is larger.
b) Upon simplification, the two terms are $(6^{32}, 2^{64})$. $6^{32} = 2^{32} \times 3^{32}$ and $2^{64} = 2^{32}2^{32}$. As $3^{32} > 2^{32}$, the first term is larger.
c) $4^{30} \times 3^{20} > 5^{40}$ b) $5^{66} < 3^{99}$ c) $2^{6018} < 3^{4012}$

**6.** a) Write $5^{11}$ as the sum of 2 squares.
b) Write $17^{17}$ as the sum of 2 squares.
c) Write $126^{301}$ as the sum of 2 cubes.
d) Write $10^{511}$ as a sum of two square numbers.
e) Write $9^{2014}$ as a sum of two cube numbers.

*Solutions:* a) $5^{11} = (1 \times 5^5)^2 + (2 \times 5^2)^2$
b) $17^{17} = (1 \times 17^8)^2 + (4 \times 17^8)^2$
c) $126^{301} = (1 \times 126^{100})^3 + (5 \times 126^{100})^3$

d) Let's do exactly the same thing here as we did for the last one. $10 = 1 + 3^2$, so
$$10^{511} = 10 \times 10^{510} = (1 + 3^2) \times (10^{255})^2 = (10^{255})^2 + 3^2 \times (10^{255})^2$$
$$= (10^{255})^2 + (3 \times 10^{255})^2$$
e) There is a slight difference here because we are using cubes, but let's use the same methodology anyway. $9 = 1 + 2^3$, so
$$9^{2014} = 9 \times 9^{2013} = (1 + 2^3) \times (9^{671})^3 = (9^{671})^3 + 2^3 \times (9^{671})^3$$
$$= (9^{671})^3 + (2 \times 9^{671})^3$$
**7.** a) Calculate $1^2 + 5^2 + 10^2 + 27^2 + 34^2$.
b) Write $2011^{2011}$ as a sum of five perfect squares.

*Solution:*
a) The answer should be 2011
b)
$$2011^{2011} = 2011^{2010} \times 2011 = (2011^{1005})^2 \times (1^2 + 5^2 + 10^2 + 27^2 + 34^2) =$$
$$= (2011^{1005})^2 \times 1^2 + (2011^{1005})^2 \times 5^2 + (2011^{1005})^2 \times 10^2 + (2011^{1005})^2 \times 27^2 + (2011^{1005})^2 \times 34^2$$
$$= (2011^{1005} \times 1)^2 + (2011^{1005} \times 5)^2 + (2011^{1005} \times 10)^2 + (2011^{1005} \times 27)^2 + (2011^{1005} \times 34)^2$$
There is a bit of clever manipulation here that the students might struggle with. If they remember everything that has been done so far, (and use it!) they might get it.

**8.** How many digits does the number $4^{16} \times 5^{25}$ have?

*Solution:* $4^{16}5^{25} = 2^{32}5^{25} = 2^7 2^{25} 5^{25} = 2^7 10^{25} = 128 \times 10^{25}$. So there are 28 digits.

**9. Digit Tricks**

*a*) Find the last digit of $5^{555}$
b) Find the last digit of $3^{2012}$
c) Find the last digit of $9^{10^{100}}$
d) Find the last digit of $2^{2^{2012}+2}$
e) The number $N = 1000! = 1000 \times 999 \times 998 \times \cdots \times 3 \times 2 \times 1$ ends in exactly how many zeroes?

*Solution:* a) This is obviously 5.

*b)* There is no way we can do this by actually figuring out what $3^{2012}$ is; there will be too many digits. But we can be clever and only use 1 digit. If we go through the first few values of $3^n$ looking only at the last digit, we get a pattern: $3, 9, 7$,1,3 ... We know that as soon as we get to 1 in our pattern that the pattern has ended. This is because the next number will be our starting number. This pattern repeats after 4 terms. So the last digit depends on what remainder 2012 has upon division by 4. Since 4 divides 2012, it's remainder is zero and $3^{2012}$'s last digit is 1.
*Note:* Some patterns don't end in 1, just look at the next example.

*c*) The pattern goes like $9$,1,9,1 ... so has a length 2. So we need to find out wether $10^{100}$ is even or not. As it is even, then $9^{10^{100}}$ has a last digit 1.

d) The pattern for the last digit of $2^n$ goes like $2$,4,8,6,2, ... so the pattern has length 4. So we need to find the remainder of $2^{2012} + 2$ upon division by 4. $2^{2012} = 4^{1006}$ is divisible by 4, so the remainder of $2^{2012} + 2$ upon division of 4 is 2. Hence, the last digit of $2^{2^{2012}+2}$ is the second term in the pattern, 4.

e) Write each of the factors of 1000! as a product of prime numbers. It suffices to find how many times 5 occurs in that product. Indeed, 10=2\times 5 and 2 will occur more times than 5 in 1000! (because of all those even numbers!), so each time a 5 occurs, it can be paired with a 2 to form a 10.

Among the numbers in the list 1, 2, ..., 1000 there are 200 multiples of 5, there are 40 multiples of 25, there are 8 multiples of 125 and 1 multiple of 625. All in all, 5 will occur 200+40+8+1=249 times.