# Shrinking Squares

This lesson plan is meant to be used in conjunction with the completing the square section of the Distributivity Lesson Plan. By now students have gotten somewhat restless with number and formula tricks, and so moving on to pictures and word problems seems like a good idea.

## Lesson Plan

- Review of the formulas for squares of sums and difference of two squares.
- Solve quadratic equations by completing of squares:

  Ex. 5 c) in the Distributivity Lesson Plan

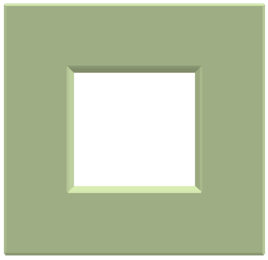
  For some students who have been taught the quadratic formula in school without explanation, it may be useful to derive the quadratic formula using the completing the square method (if they're curious).
- Practice quadratic equations with Shrinking Squares problems.
- Leave some of the questions for homework: For example Q6 and 7 or 6 and 8.
- Return to section 6 in Distributivity problems.

## Resources

- Calculators may prove useful.
- 1 Activities sheet per student.
- Folding paper might be fun: 3-4 square pieces per student.

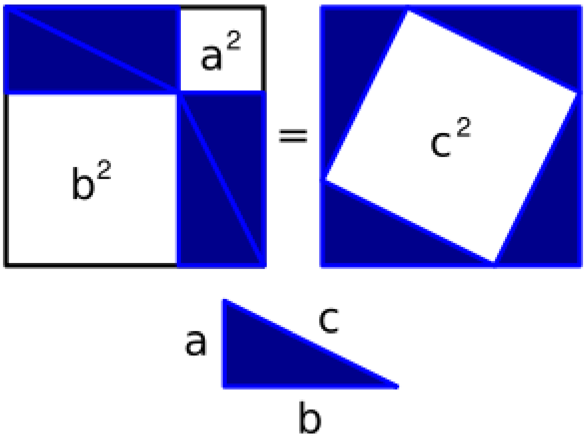
# Shrinking Squares

## 1. Insert portrait here.



A square photo is surrounded by a 4 cm wide green frame. Although it may not look like it, the area of the actual photo is only one third of the area of the green frame. How large is the photo?

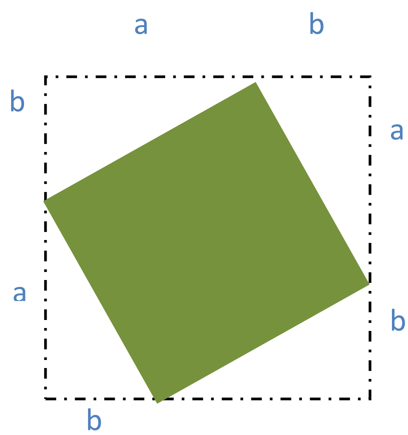
## 2. Pythagoras' Theorem:




Explain how the picture on the left proves phythagoras theorem.

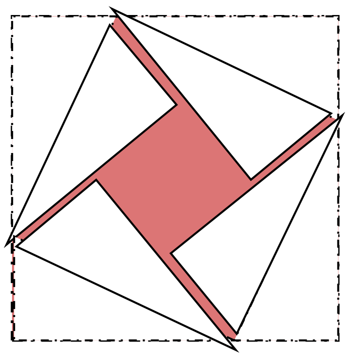
## 3. Cutting

Start with a square of side 10 cm.
Cut the corners symmetrically like in the picture:
How big should the sides of the cut-out corners be
so that the remaining green area is exactly 84 $cm^2$?



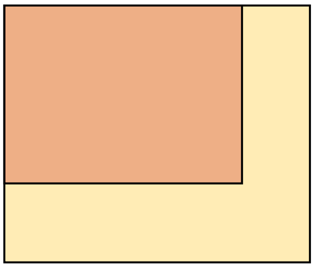

## 4. Folding

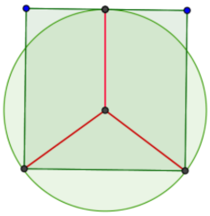
Start with a square of side 10 cm.
Fold the corners symmetrically like in the picture:
How big should be the sides of the folded corners
so that the interior area is a square of area 64 $cm^2$?



## 5. Extended comfort

Your house consists of one rectangular room of area 120 $m^2$. You break two walls and extend one side by 2 m, and the other side by 3 m, thus gaining an area of 60 $m^2$. How big were the sides of the house originally?





## 6. Love and Peace.

The picture here is symmetrical, it is made of a circle of radius 3 cm and a square. Find the side of the square.

## 7. Love and Finances

Alice bought her wedding gown fabric for RWF 24,000. If she had bought the fabric from a different shop which charged 400 RWF more per meter, she would have got 3 meters less for the same amount money. How many meters did she purchase?

## 8. Racing

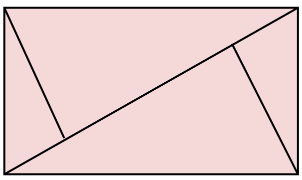
Ralph is standing along Felicity Road which heads straight East. If you go 9 km East, make a left turn and travel 6 km North, you will find Pamela with her mountain bike. At the exact same time that Ralph begins running eastward along the road at 6 km per hour, Pamela begins biking in a straight line at 10 km per hour. Pamela's direction is chosen so that she will meet Ralph exactly when she reaches Felicity Road. How long does it take Pamela to reach Ralph?

## 9. Final Folding

A piece of paper can be folded into a rectangular envelope of sides 8 cm and 6 cm, like in the picture.

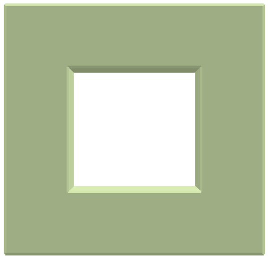
a) Draw the piece of paper as it looked before the folding.

b) Find the length of the sides of the original piece of paper, and where exactly you should fold to get the envelope.



# Shrinking Squares Solutions

## 1. Insert portrait here.



A square photo is surrounded by a 4 cm wide green frame. Although it may not look like it, the area of the actual photo is only one third of the area of the green frame. How large is the photo (that is, the area inside the frame)?

*Solution:*

Let $x$ be the side length of the photo. We know photo : frame=1:3.

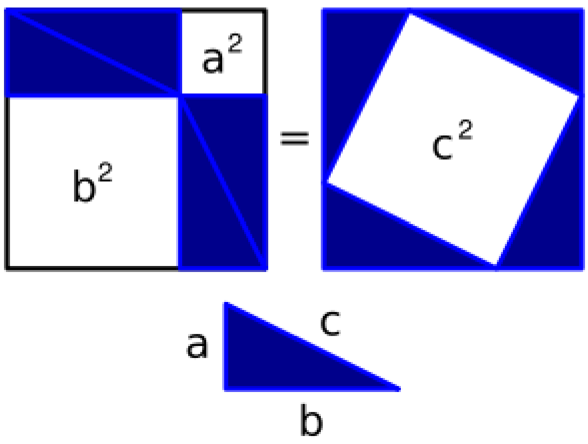
Another way to say this is that the area of the photo is ¼ of the total area of photo+frame.

This total area is $(x+8)^2$, the area of the photo is $x^2$, so

$$(x+8)^2 = 4x^2$$

Since we're dealing with positive numbers we can just deduce $x+8=2x$ so $x=8$.

## 2. Pythagoras' Theorem:



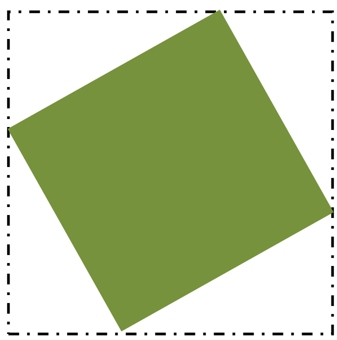

Pythagoras' Theorem:

If a triangle has a right angle, then the square of the side opposite the right angle is equals the sum of squares of the other sides:

$$c^2 = a^2 + b^2$$

The side opposite the right angle is called hypotenuse.

## 3. Cutting



Start with a square of side 10 cm.
Cut the corners symmetrically like in the picture:
How big should be the sides of the cut-out corners so that the remaining area to be exactly 84 cm$^2$?

*Solution:*

Let $x$ be one of the sides of the corner triangle adjoining the corner, so that the other side is $10-x$.

Using the diagram in Pythagoras' Theorem to get the formula for the green square:

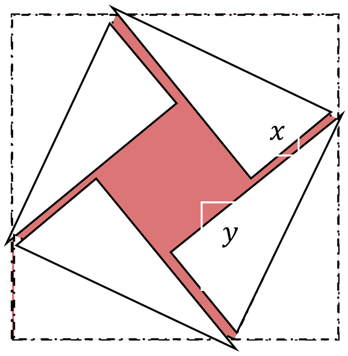
$$x^2 + (10 - x)^2 = 84.$$

Another way to find this is $100 - 2x(10 - x) = 84.$

In either case we simplify to $2x^2 - 20x + 100 = 84,$ and further to $x^2 - 10x + 8 = 0.$

Complete the square: $(x - 5)^2 - 17 = 0$ so $x = 5 - \sqrt{17}$ or $x = 5 + \sqrt{17}$. These are the sides of the corner triangle which are next to the corner. The hypothenuse is $\sqrt{84}$.

## 4. Folding




Start with a square of side 10 cm.
Fold the corners symmetrically like in the picture:
How big should be the sides of the folded corners so that the interior area is a square of area 64 cm$^2$?

*Solution I:*

Let $x$ be one of the sides of the corner triangle adjoining the corner, so that the other side is $y = 10 - x$. When the two sides get folded, their difference is $y - x = 8\ cm$.

So $10 - x - x = 8\ cm$. Solving we get $x = 1\ cm$ and $y = 9\ cm$.

The hypothenuse is $\sqrt{1^2 + 9^2} = \sqrt{82}$.

*Solution II:*

Let $x$ be one of the sides of the corner triangle adjoining the corner, so that the other side is $10 - x$.
Now the equation for the area of the interior square is

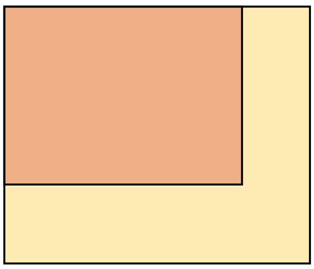
$$100 - 4x(10 - x) = 64,$$

Which after working step-by-step simplifies to $x^2 - 10x + 9 = 0$.

By completing the square and using the difference of squares, factor into $(x - 1)(x - 9) = 0$.

So $x = 1$ and $x = 9$ are the two sides next to the right angle. The hypothenuse is $\sqrt{1^2 + 9^2} = \sqrt{82}$.

## 5. Extended comfort



Your seaside villa consists of one rectangular room of area 120 m$^2$. You break two walls and extend one side by 2 m, and the other side by 3 m, thus gaining an area of 60 m$^2$. What were the length and width of the villa originally?

*Solution:* This is slightly different from the other problems because it's a "completing the rectangle." Let $x$ and $y$ be the two sides of the original villa, so that the sides of the new villa are $x + 2$ and $y + 3$. The old villa had an area of $xy = 120$, while the new villa has an area of

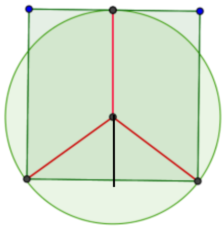
$$(x + 2)(y + 3) = xy + 2y + 3x + 6.$$

The newly added area is $2y + 3x + 6 = 60$.

We get a system of equations: $xy = 120$ and $2y + 3x = 54$. Solve for $y = 27 - \frac{3x}{2}$ and substitute in the first equation: $x(27 - \frac{3x}{2}) = 120$ so $54x - 3x^2 = 240$. Simplify by $-3$ so $x^2 - 18x = -80$

Complete the square: $x^2 - 18x + 80 = (x - 9)^2 - 81 + 80 = (x - 9)^2 - 1 = (x - 10)(x - 8)$.

We get $x = 10$ and $y = 12$ or $x = 8$ and $y = 15$.



## 6. Love and Peace.

The picture here is symmetrical, it is made of a circle of radius 3 cm and a square. Find the side of the square.

*Solution:* Let $x$ be the length of the square. We extend the vertical line through the middle of the square and apply Pythagora's theorem in one of the two small triangles thus formed:

$$(x - 3)^2 + \left(\frac{x}{2}\right)^2 = 3^2.$$

Simplify to $x^2 - 6x + 9 + \frac{x^2}{4} = 9$ so $\frac{5x^2}{4} - 6x = 0$ so $x\left(\frac{5x}{4} - 6\right) = 0$. Since $x \neq 0$,

It follows that $\frac{5x}{4} = 6$ so $x = 4.8$ cm. (Yes, in my experience – you do need to go through each of the steps above and explain them carefully to the audience).

## 7. Love and Finances

Alice bought her wedding gown fabric for EU 240. If she had bought the fabric from a different shop which charged 4 EU more per meter, she would have got 3 meters less for the same amount money. How many meters did she purchase?

*Solution:* Let $x$ be the length of the fabric roll she bought. So the price per meter she paid was $\frac{240}{x}$. If she had bought the fabric from a different shop she would have paid $\frac{240}{x} + 4$ per metre and bought $x - 3$ meters. All in all $\left(\frac{240}{x} + 4\right)(x - 3) = 240$.

Simplify $\frac{240+4x}{x}(x - 3) = 240$, or $\frac{60+x}{x}(x - 3) = 60$, or $(60 + x)(x - 3) = 60x$.

Simplify to $x^2 - 3x - 180 = 0$. With completing the square and difference of two squares formulas:

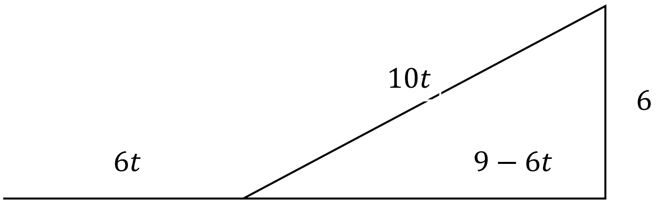
$$(x - 15)(x + 12) = 0.$$

The only positive solution is $x = 15$ meters for a gown!

## 8. Racing I

Ralph is standing along Felicity Road which heads straight East. If you go 9 km East, make a left turn and travel 6 km North, you will find Pamela with her mountain bike. At the exact same time that Ralph begins running eastward along the road at 6 km per hour, Pamela begins biking in a straight line at 10 km per hour. Pamela's direction is chosen so that she will meet Ralph exactly when she reaches Felicity Road. How long does it take Pamela to reach Ralph?

*Solution:* Let $t$ be the time it takes Pamela and Ralph to meet. Ralph will move on a straight line exactly $6t$ meters. Pamela will move along the hypothenuse of the triangle here:




By Pythagoras' theorem we get $\mathbf{100t^2 = (9-6t)^2 + 36}$. Solving this $100t^2 = 36t^2 - 108t + 117$ so

$$64t^2 + 108t - 117 = (8t)^2 + 2\cdot 8t\cdot\frac{108}{16} - 117 = \left(8t + \frac{27}{4}\right)^2 - \left(\frac{27}{4}\right)^2 - 117 = \left(8t + \frac{27}{4}\right)^2 - \frac{2601}{4}$$

$$= \left(\mathbf{8t} + \frac{\mathbf{27}}{\mathbf{4}} - \frac{\mathbf{51}}{\mathbf{4}}\right)\left(\mathbf{8t} + \frac{\mathbf{27}}{\mathbf{4}} + \frac{\mathbf{51}}{\mathbf{4}}\right) = (\mathbf{8t} - \mathbf{6})(\mathbf{8t} + \mathbf{19}).$$

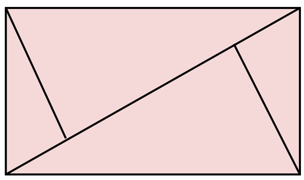
Hence $t = \frac{3}{4}$ (time should be positive).

## 9. Final Folding

A piece of paper can be folded into a rectangular envelope of sides 8 cm and 6 cm, like in the picture.

a) Draw the piece of paper as it looked before the folding.

b) Find the lengths of the sides of the original piece of paper, and where you need to fold.



*Solution :* First we can calculate the diagonal $|GE|$ of the envelope by applying Pythagoras' theorem in triangle $\Delta HEG$.

We have $|HE| = 6$ cm and $|HG| = 8$ cm so we get $|EG| = 10$ cm.

By symmetry

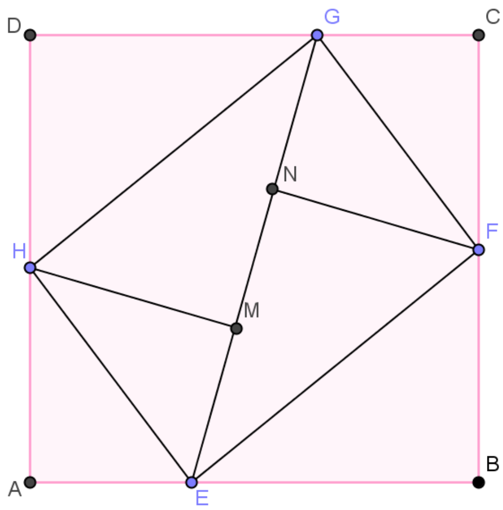
$$|AE| = |EM| = |NG| = |GC| = x$$

Mark these down in the diagram and note

$$|BE| = |EN| = |MG| = |GD| = 10 - x$$

In particular, adding up we get

$$|AB| = |EG| = |CD| = 10\ cm.$$



Let $|HM| = |FN| = y$.

*Method I:* The total area of the envelope can be calculated as

$$y \cdot 10 = 6 \cdot 8 \, cm^2.$$

Hence $y = 4.8\ cm$ and so |AD|=|BC| $= 2y = 9.6$ cm. It remains to apply Pythagoras' theorem in triangle HMG to find $x = \sqrt{6^2 - 4.8^2} = 3.6\ cm$.

*Method II:* By Pythagoras' theorem in triangles HMG and HME:

$$y^2 + x^2 = 6^2$$

$$y^2 + (10 - x)^2 = 8^2$$

Subtract: $(10 - x)^2 - x^2 = 64 - 36 = 28$ so $100 - 20x = 28$ hence $x = 3.6$ cm.

We can now find $|AH| = |HM| = |HD| = |FC| = |FN|$ from Pythagoras' theorem in triangle HME:

$$y^2 = 6^2 - 3.6^2 = 4.8^2$$

We found $y = 4.8$ and so |AD|=|BC| $= 2y = 9.6$ cm

The original piece of paper was very nearly a square but not quite so!